2.3 Curve curvature

Definition 2.3.1 (Unit tangent and normal vector). *Let* $\mathbf{r}(t)$ *be a regular parametrized curve.*

1. *The* **unit tangent vector** *to the curve at* $\mathbf{r}(t)$ *is defined by*

$$\boxed{\mathbf{T}(t) = \frac{\mathbf{r}'(t)}{\|\mathbf{r}'(t)\|}.}$$

In particular if $\mathbf{r}(s)$ *is an arc length parametrization, then*

$$\mathbf{T}(s) = \mathbf{r}'(s).$$

2. *Suppose* $\mathbf{T}'(t) \neq 0$. *We define the* **unit normal vector** *to the curve at* $\mathbf{r}(t)$ *by*

$$\boxed{\mathbf{N}(t) = \frac{\mathbf{T}'(t)}{\|\mathbf{T}'(t)\|}.}$$

In particular if $\mathbf{r}(s)$ *is an arc length parametrization, then*

$$\mathbf{N}(s) = \frac{\mathbf{T}'(s)}{\|\mathbf{T}'(s)\|} = \frac{\mathbf{r}''(s)}{\|\mathbf{r}''(s)\|}.$$

$$r'(t) = \overbrace{\|r'(t)\|}^{\text{speed}}\ \overbrace{\frac{r'(t)}{\|r'(t)\|}}^{\text{direction}}$$

$$= \|r'(t)\|\, T(t)$$

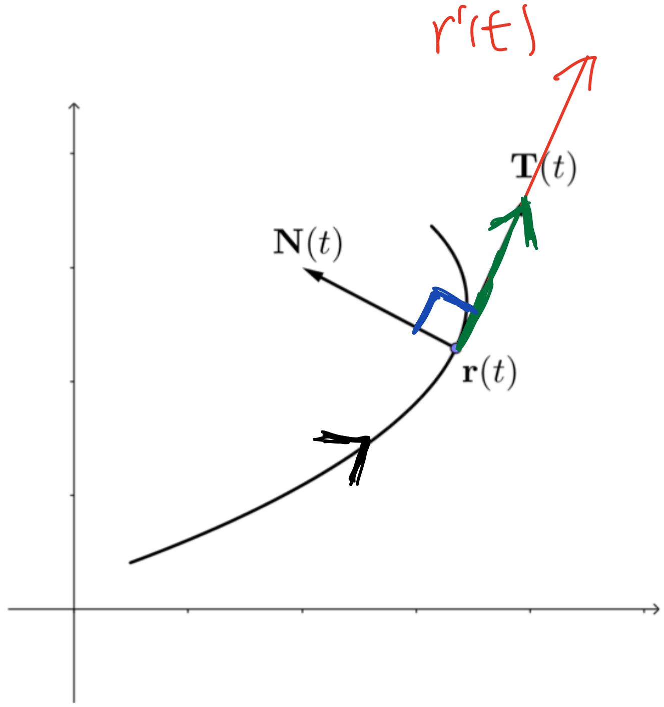


Figure 7: Unit tangent and unit normal vector

<u>Prop</u> $T(t) \perp N(t)$

$$\langle T(t), N(t)\rangle = \left\langle T(t), \frac{T'(t)}{\|T'(t)\|}\right\rangle$$

$$= \frac{1}{\|T'(t)\|}\langle T(t), T'(t)\rangle$$

$$\overset{?}{=} 0 \quad *$$

*

$$\|T\| \equiv 1$$

$$\langle T, T\rangle = \|T\|^2 \equiv 1$$

$$\frac{d}{dt}\langle T, T\rangle = 0$$

$$2\langle T, T'\rangle = 0$$

Proposition 2.3.2. *Let* $\mathbf{r}(t)$ *be a regular parametrized curve and* $\mathbf{N}(t)$ *be the unit normal vector. We have*

1. $\displaystyle \frac{d}{dt}\|\mathbf{r}'\| = \frac{\langle \mathbf{r}', \mathbf{r}''\rangle}{\|\mathbf{r}'\|}$

2. $\displaystyle \mathbf{T}' = \frac{\mathbf{r}''}{\|\mathbf{r}'\|} - \frac{\langle \mathbf{r}', \mathbf{r}''\rangle}{\|\mathbf{r}'\|^3}\mathbf{r}'$

Pf ① L.H.S. $= \frac{d}{dt}\sqrt{\langle r', r'\rangle}$

$$= \frac{2\langle r', r''\rangle}{2\sqrt{\langle r', r'\rangle}}$$

$$= \frac{\langle r', r''\rangle}{\sqrt{\langle r', r'\rangle}}$$

$$\langle r', r'\rangle' = \langle r'', r'\rangle + \langle r', r''\rangle$$

② $T = \frac{r'}{\|r'\|}$

$$T' = \frac{\|r'\| r'' - \|r'\|' r'}{\|r'\|^2}$$

$$= \frac{\|r'\| r'' - \frac{\langle r', r''\rangle}{\|r'\|} r'}{\|r'\|^2}$$

$=$ R.H.S.

Definition 2.3.3 (Curve curvature)**.** *Let* $\mathbf{r}(t)$ *be a regular parametrized curve and* $\mathbf{T}(t)$ *be the unit tangent to the curve at* $\mathbf{r}(t)$. *Then the* **curvature** *of the curve at* $\mathbf{r}(t)$ *is*

$$\kappa(t) = \frac{\|\mathbf{T}'(t)\|}{\|\mathbf{r}'(t)\|}.$$

← rate of change of direction

← rate of change of displacement / speed

In particular if $\mathbf{r}(s)$ *is an arc length parametrized curve, the curvature is*

$$\kappa(s) = \|\mathbf{T}'(s)\|$$

← change of direction with respect to arclength

Proposition 2.3.4. *Let* $\mathbf{r}(t)$ *be a regular parametrized curve. Then the curvature satisfies* $\kappa(t) = 0$ *for any* $a \le t \le b$ *if and only if* $\mathbf{r}(t)$ *is a straight line segment joining* $\mathbf{r}_0$ *and* $\mathbf{r}_1$.

$r_0 = r(a) \quad r_1 = r(b)$

$\kappa(t) \equiv 0 \iff$ straight line

Pf ($\Rightarrow$) Suppose $\kappa(t) \equiv 0$

$$\Rightarrow \frac{\|T'(t)\|}{\|r'(t)\|} \equiv 0$$

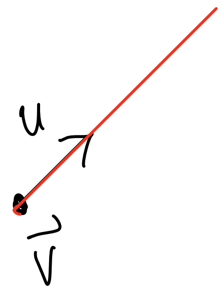
$$\Rightarrow T'(t) \equiv 0$$

$\Rightarrow T(t) \equiv \vec{u}$ for some unit vector

$$\Rightarrow \frac{r'(t)}{\|r'(t)\|} \equiv \vec{u}$$

$$\Rightarrow r'(t) = \|r'(t)\|\,\vec{u}$$

$$\Rightarrow r(t) = \int \|r'(t)\|\,dt\,\vec{u} + \vec{v}$$



($\Leftarrow$) Suppose $r(t)$ is straight line

$$r(t) = \vec{v} + \alpha(t)\,\vec{u}$$

$$r'(t) = \alpha'(t)\,\vec{u} \neq \vec{0}$$

$\alpha(t)$ increasing function $\Rightarrow \alpha' > 0$

$$T = \frac{r'}{\|r'\|} = \frac{\alpha'(t)\,\vec{u}}{\|\alpha'(t)\,\vec{u}\|}$$

$$= \frac{\alpha'(t)}{|\alpha'(t)|\,\cancel{\|u\|}}\,\vec{u} = \vec{u}$$

$$\kappa(t) = \frac{\|T'(t)\|}{\|r'(t)\|} = \frac{\|\vec{0}\|}{\|r'(t)\|} = 0$$

Proposition 2.3.5 (Formulas for curvature). *Let $\mathbf{r}(t)$ be a regular parametrized curve.*

1. *Suppose $\mathbf{r}(t) = (x(t), y(t))$ is a plane curve. Then*
$$\kappa(t) = \frac{|x'y'' - x''y'|}{(x'^2 + y'^2)^{\frac{3}{2}}}. \quad = \left.\begin{vmatrix} x' & y' \\ x'' & y'' \end{vmatrix} \middle/ \|r'\|^3 \right.$$

2. *Suppose $\mathbf{r}(t) = (x(t), y(t), z(t))$ is a space curve. Then*
$$\kappa(t) = \frac{\|\mathbf{r}' \times \mathbf{r}''\|}{\|\mathbf{r}'\|^3}.$$

Rmk ① is special case of ②

Regard $r(t) = (x, y) = (x, y, 0)$

①
$$\kappa = \frac{\|T'\|}{\|r'\|} = \frac{1}{\|r'\|}\left\|\frac{\|r'\|^2 r'' - \langle r', r''\rangle r'}{\|r'\|^3}\right\|$$

$$\frac{d}{dt}\|\mathbf{r}'\| = \frac{\langle \mathbf{r}', \mathbf{r}''\rangle}{\|\mathbf{r}'\|}$$

$$\mathbf{T}' = \frac{\mathbf{r}''}{\|\mathbf{r}'\|} - \frac{\langle \mathbf{r}', \mathbf{r}''\rangle}{\|\mathbf{r}'\|^3}\mathbf{r}'$$

$r' = (x', y') \quad r'' = (x'', y'')$

$\langle r', r''\rangle = x'x'' + y'y''$

$$\|r'\|^2 r'' - \langle r', r''\rangle r' = (x'^2 + y'^2)(x'', y'') - (x'x'' + y'y'')(x', y')$$
$$= [(x')^2 x'' + (y')^2 x'' - (x')^2 x'' - x'y'y'', \; \cdots \;]$$

$$= [-y'(x'y''-x''y'), x'(x'y''-x''y')]$$

$$= (x'y''-x''y')(-y', x')$$

$$K = \frac{1}{\|r'\|^4} \|(x'y''-x''y')(-y', x')\|$$

$$= \frac{1}{\|r'\|^4} |x'y''-x''y'| \sqrt{(-y')^2+(x')^2}$$

2. *Suppose* $\mathbf{r}(t) = (x(t), y(t), z(t))$ *is a space curve. Then*

$$\kappa(t) = \frac{\|\mathbf{r}' \times \mathbf{r}''\|}{\|\mathbf{r}'\|^3}.$$

$$\mathbf{T}' = \frac{\mathbf{r}''}{\|\mathbf{r}'\|} - \frac{\langle \mathbf{r}', \mathbf{r}'' \rangle}{\|\mathbf{r}'\|^3}\mathbf{r}'$$

$$\kappa = \frac{\|\mathbf{T}'\|}{\|\mathbf{r}'\|} = \left\| \frac{\|\mathbf{r}'\|^2\mathbf{r}'' - \langle \mathbf{r}', \mathbf{r}'' \rangle \mathbf{r}'}{\|\mathbf{r}'\|^4} \right\|.$$

See lecture note

Theorem 2.3.6. *Suppose* $\mathbf{r}(s)$ *is an arc length parametrized curve. Then*

1. $\kappa(s) = \|\mathbf{r}''(s)\|$
2. $\mathbf{T}'(s) = \kappa(s)\mathbf{N}(s)$

Pf ① $T(s) = \dfrac{r'(s)}{\|r'(s)\|} = r'(s)$

$$\kappa(s) = \frac{\|T'(s)\|}{\|r'(s)\|} = \|T'(s)\| = \|r''(s)\|$$

② $$N(s) = \frac{T'(s)}{\|T'(s)\|}$$

$$T'(s) = \|T'(s)\| \, N(s)$$

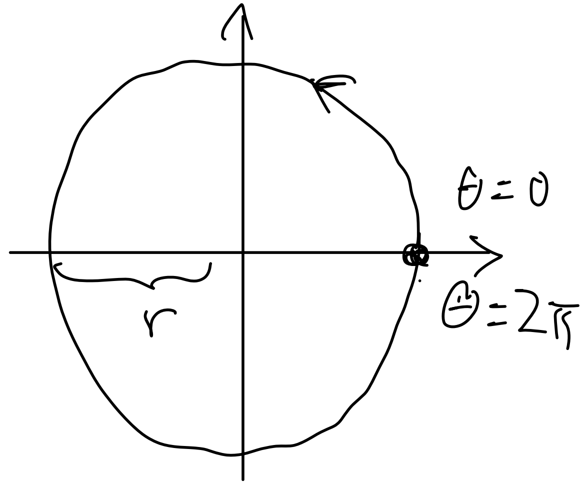
$$= \kappa(s) \, N(s)$$

Example 2.3.7 (Circle). *Let* $\mathbf{r}(\theta) = (\overset{x}{\overset{\parallel}{r\cos\theta}}, \overset{y}{\overset{\parallel}{r\sin\theta}})$, $0 < \theta < 2\pi$, *be the circle of radius* $r > 0$ *centered at the origin. Then*

$$x(\theta) = x = r\cos\theta \qquad y(\theta) = y = r\sin\theta$$

$$x' = -r\sin\theta \qquad y' = r\cos\theta$$

$$x'' = -r\cos\theta \qquad y'' = -r\sin\theta$$



$$K = \frac{|x'y'' - x''y'|}{\left((x')^2 + (y')^2\right)^{3/2}} = \frac{\left|\det\begin{bmatrix} x' & y' \\ x'' & y'' \end{bmatrix}\right|}{\|\mathbf{r}'\|^3}$$

$$= \frac{|r^2\sin^2\theta - (-r^2\cos^2\theta)|}{\left(r^2(\sin^2\theta + \cos^2\theta)\right)^{3/2}}$$

$$= \frac{|r^2|}{|r^2|^{3/2}} = \frac{1}{r}$$

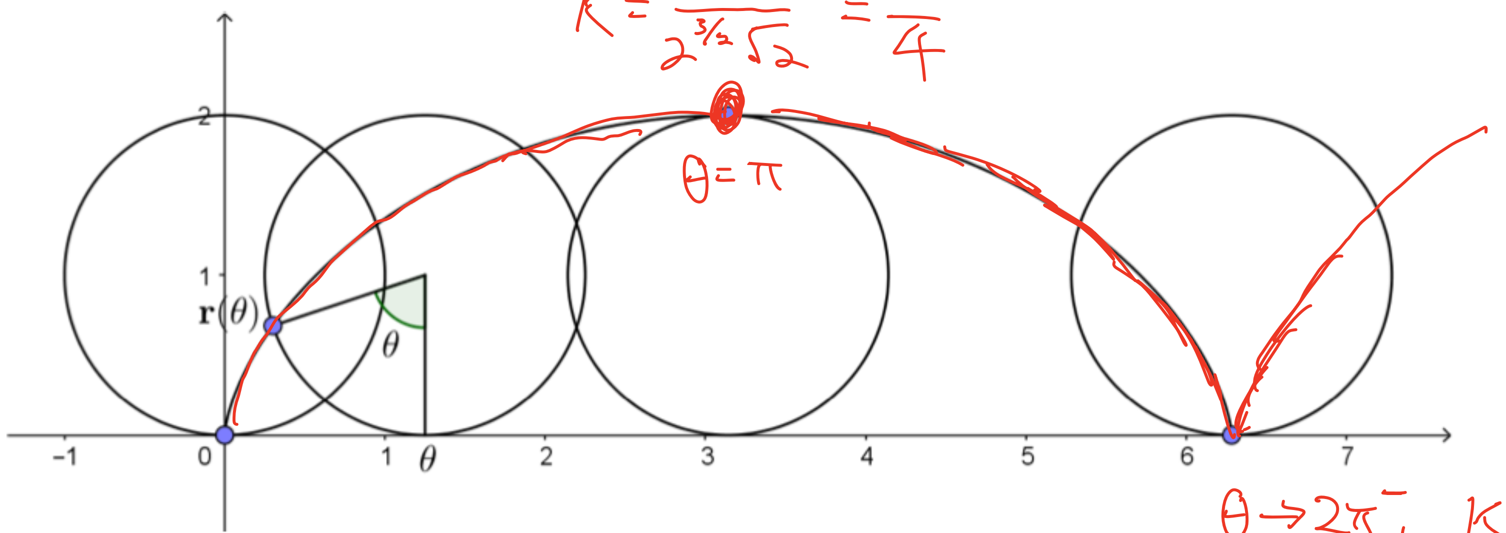
Example 2.3.8 (Cycloid). *The* **cycloid** *is the curve parametrized by*

$$\mathbf{r}(\theta) = (\theta - \sin\theta, 1 - \cos\theta), \ \textit{for } \theta \in (0, 2\pi).$$

$$\begin{cases} \mathbf{r}'(\theta) = (1 - \cos\theta, \sin\theta) \\ \mathbf{r}''(\theta) = (\sin\theta, \cos\theta) \end{cases}$$

$$\begin{aligned} \kappa(\theta) &= \frac{|x'y'' - x''y'|}{(x'^2 + y'^2)^{\frac{3}{2}}} \\ &= \frac{|(1 - \cos\theta)\cos\theta - \sin\theta\sin\theta|}{((1 - \cos\theta)^2 + (-\sin\theta)^2)^{\frac{3}{2}}} \\ &= \frac{1 - \cos\theta}{(2 - 2\cos\theta)^{\frac{3}{2}}} \\ &= \frac{1}{2^{\frac{3}{2}}\sqrt{1 - \cos\theta}}. \end{aligned}$$

$\kappa = \frac{1}{2^{3/2}\sqrt{2}} = \frac{1}{4}$



$\theta \to 2\pi^-, \ \kappa \to \infty$

Figure 4: Cycloid

Example 2.3.9 (Helix). *Let $a, b > 0$ be constants. The space curve $\mathbf{r}(\theta) = (a\cos\theta, a\sin\theta, b\theta)$, $\theta \in \mathbb{R}$, is called a* **helix**. *Then*

$$r' = r'(\theta) = (-a\sin\theta, a\cos\theta, b) \qquad \|r'\| = \sqrt{a^2+b^2}$$

$$r'' = (-a\cos\theta, -a\sin\theta, 0)$$

$$r' \times r'' = \begin{vmatrix} i & j & k \\ -a\sin\theta & a\cos\theta & b \\ -a\cos\theta & -a\sin\theta & 0 \end{vmatrix}$$

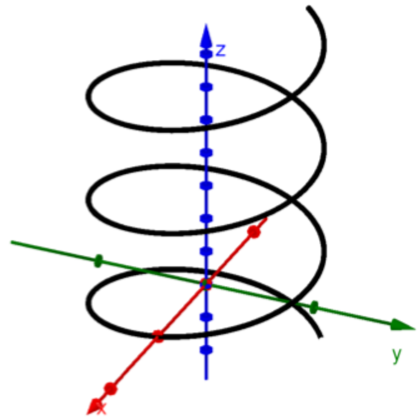


Figure 5: Helix

$$\|r' \times r''\| = \sqrt{\begin{vmatrix} a\cos\theta & b \\ -a\sin\theta & 0 \end{vmatrix}^2 + \begin{vmatrix} -a\sin\theta & b \\ -a\cos\theta & 0 \end{vmatrix}^2 + \begin{vmatrix} -a\sin\theta & a\cos\theta \\ -a\cos\theta & -a\sin\theta \end{vmatrix}^2}$$

$$= \sqrt{(ab\sin\theta)^2 + (ab\cos\theta)^2 + a^4}$$

$$= \sqrt{a^4 + a^2b^2} = a\sqrt{a^2+b^2}$$

$$K(\theta) = \frac{\|r' \times r''\|}{\|r'\|^3} = \frac{a\sqrt{a^2+b^2}}{(a^2+b^2)^{3/2}} = \frac{a}{a^2+b^2}$$

Proposition 2.3.10 (Curvature of graphs of functions).

1. *(Rectangular coordinates): The curvature of the curve given by the graph of function $y = f(x)$ in rectangular coordinates is*

$$\kappa(x) = \frac{|f''|}{(1 + f'^2)^{\frac{3}{2}}}.$$

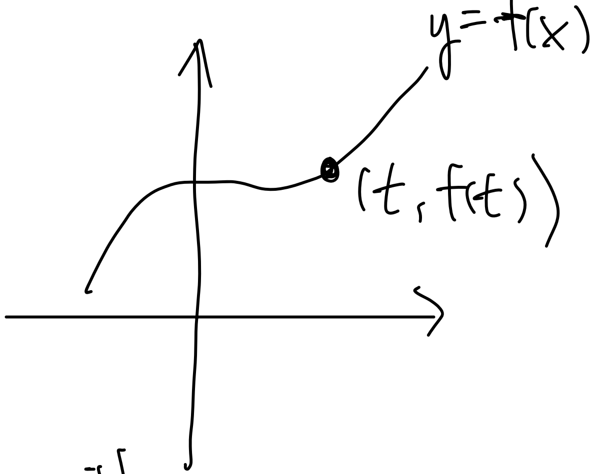
2. *(Polar coordinates): The curvature of the curve given by the graph of function $r = r(\theta)$ in polar coordinates is*

$$\kappa(\theta) = \frac{|r^2 + 2r'^2 - rr''|}{(r^2 + r'^2)^{\frac{3}{2}}}.$$

1 $x = t$ $\quad y = f(t)$

$x' = 1$ $\quad y' = f'$

$x'' = 0$ $\quad y'' = f''$



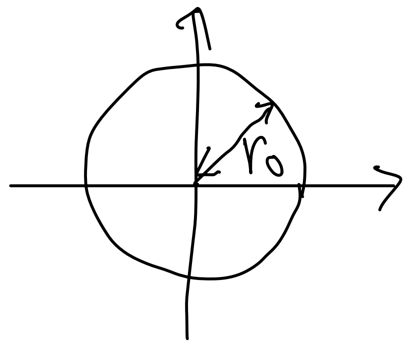
$$K = \frac{\left|\det\begin{bmatrix} x' & y' \\ x'' & y'' \end{bmatrix}\right|}{\left[(x')^2 + (y')^2\right]^{3/2}} = \frac{\left|\det\begin{bmatrix} 1 & f' \\ 0 & f'' \end{bmatrix}\right|}{\left(1 + (f')^2\right)^{3/2}} = \frac{|f''|}{\left[1 + (f')^2\right]^{3/2}}$$

2 $x = r(\theta)\cos\theta$

$y = r(\theta)\sin\theta$

Exercise / Note

eg $r = r(\theta) = r_0$



$r = r(\theta) = r_0$

$r' = r'' = 0$

$$K = \frac{|r_0^2|}{|r_0^2|^{3/2}} = \frac{1}{r_0}$$

Example 2.3.11 (Catenary). *The* **catenary** *is the curve given by the graph of the function $y = \cosh x$. Show that the curvature of the catenary is*

$\overset{\shortparallel}{f(x)}$

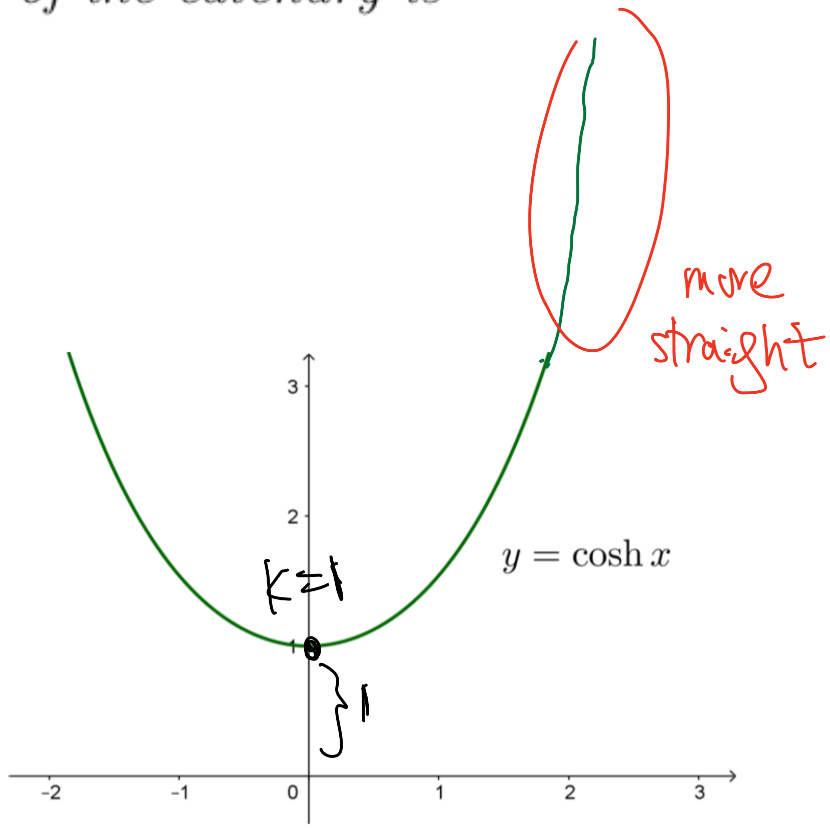
$$\kappa = \frac{1}{\cosh^2 x}.$$

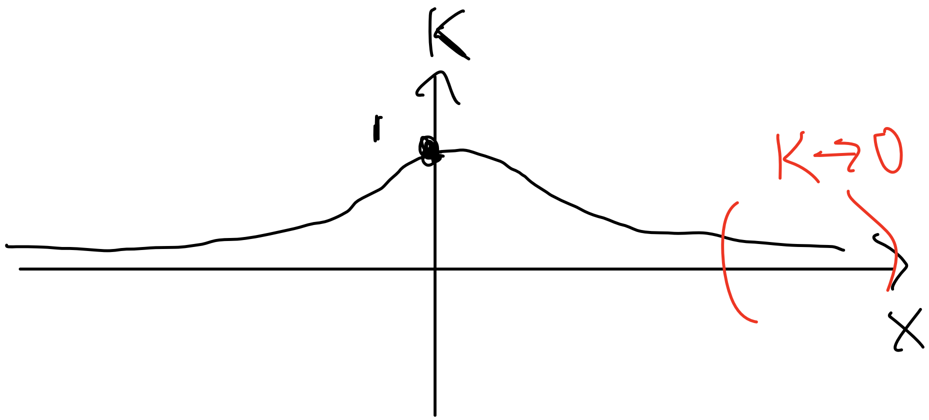
Proof. Observe that

$$\begin{cases} f'(x) = \sinh x, \\ f''(x) = \cosh x. \end{cases}$$

By Proposition $\boxed{2.3.10}$, the curvature of the catenary is

$$\begin{aligned} \kappa &= \frac{|f''|}{(1+f'^2)^{\frac{3}{2}}} \\ &= \frac{\cosh x}{(1+\sinh^2 x)^{\frac{3}{2}}} \\ &= \frac{\cosh x}{(\cosh^2 x)^{\frac{3}{2}}} \\ &= \frac{1}{\cosh^2 x} \end{aligned}$$



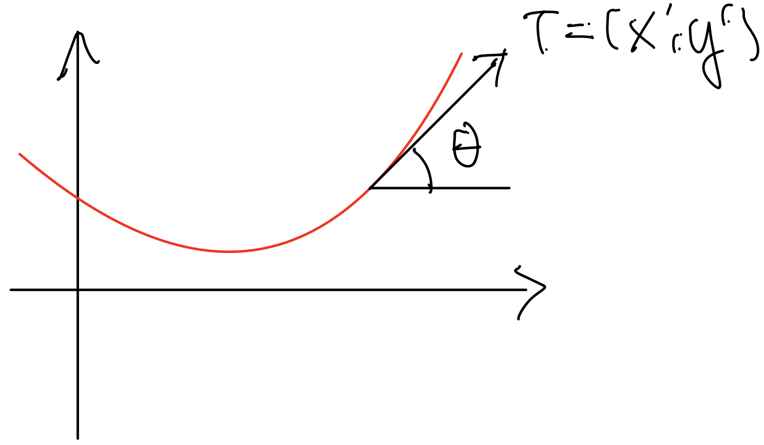


Parametrized Curve	Arc length	Curvature
Plane curve $\mathbf{r}(t) = (x(t), y(t)),$ $a < t < b$	$\int_a^b \lVert\mathbf{r}'\rVert dt$	$\kappa(t) = \dfrac{\lvert x'y'' - x''y'\rvert}{(x'^2 + y'^2)^{\frac{3}{2}}}$
Space curve $\mathbf{r}(t) = (x(t), y(t), z(t)),$ $a < t < b$	$\int_a^b \lVert\mathbf{r}'\rVert dt$	$\kappa(t) = \dfrac{\lVert\mathbf{r}' \times \mathbf{r}''\rVert}{\lVert\mathbf{r}'\rVert^3}$
Arc length parametrized curve $\mathbf{r}(s)$ with $\lVert\mathbf{r}(s)\rVert = 1$ $a < s < b$	$b - a$	$\kappa(s) = \lVert\mathbf{r}''(s)\rVert$
Circle $\mathbf{r}(\theta) = (r\cos\theta, r\sin\theta),$ $0 < \theta < 2\pi$	$2\pi r$	$\kappa = \dfrac{1}{r}$
Cycloid $\mathbf{r}(\theta) = (\theta - \sin\theta, \cos\theta),$ $\theta \in (0, 2\pi)$	8	$\dfrac{1}{2^{\frac{3}{2}}\sqrt{1-\cos\theta}}$
Helix $\mathbf{r}(\theta) = (a\cos\theta, a\sin\theta, b\theta),$ $0 < \theta < 2\pi$	$2\pi\sqrt{a^2+b^2}$	$\dfrac{a}{a^2+b^2}$
Graph of function $y = f(z)$ in rectangular coordinates $\mathbf{r}(t) = (t, f(t)),$ $a < t < b$	$\int_a^b \sqrt{1+f'^2}dx$	$\dfrac{\lvert f''\rvert}{(1+f'^2)^{\frac{3}{2}}}$
Graph of function $r = r(\theta)$ in polar coordination $\mathbf{r}(\theta) = (r\cos\theta, r\sin\theta),$ $\alpha < \theta < \beta$	$\int_\alpha^\beta \sqrt{r^2+r'^2}d\theta$	$\dfrac{\lvert r^2 + 2r'^2 - rr''\rvert}{(r^2+r'^2)^{\frac{3}{2}}}$

12:09

Proposition 2.3.12. *Let* $\mathbf{r}(s)$ *be an arc length parametrized plane curve and* $\theta(s)$ *be the angle between* $\mathbf{T}$ *and positive* x*-axis. Then*

$$\kappa(s) = \left|\frac{d\theta}{ds}\right|.$$



$r = (x, y) \qquad T = \frac{(x', y')}{\|(x', y')\|} = (x', y')$

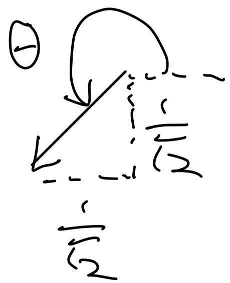
$\tan\theta = \frac{y'}{x'} \qquad \theta = \arctan\left(\frac{y'}{x'}\right)$

$$\frac{d\theta}{ds} = \frac{1}{1+\left(\frac{y'}{x'}\right)^2}\,\frac{x'y''-x''y'}{(x')^2} \qquad -\frac{\pi}{2} < \theta < \frac{\pi}{2} \qquad \frac{d}{dz}\arctan z = \frac{1}{1+z^2}$$

$$= \frac{x'y''-x''y'}{(x')^2+(y')^2} = x'y''-x''y'$$

Rmk arctan is not good for $\theta \notin \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

$$\kappa = \frac{|x'y''-x''y'|}{\left[(x')^2+(y')^2\right]^{3/2}} = |x'y''-x''y'|$$



$$\arctan\frac{y'}{x'} = \frac{\pi}{4} \neq \frac{5\pi}{4}$$

Another way: good for any $\theta \in \mathbb{R}$

$$x' = \cos\theta \qquad y' = \sin\theta$$

$$x'' = -\sin\theta \frac{d\theta}{ds} \qquad y'' = \cos\theta \frac{d\theta}{ds}$$

$$\kappa = \frac{|x'y'' - x''y'|}{\left[(x')^2 + (y')^2\right]^{3/2}}$$

$$= \frac{\left|\cos^2\theta \frac{d\theta}{ds} - (-\sin^2\theta)\frac{d\theta}{ds}\right|}{(\cos^2\theta + \sin^2\theta)^{3/2}}$$

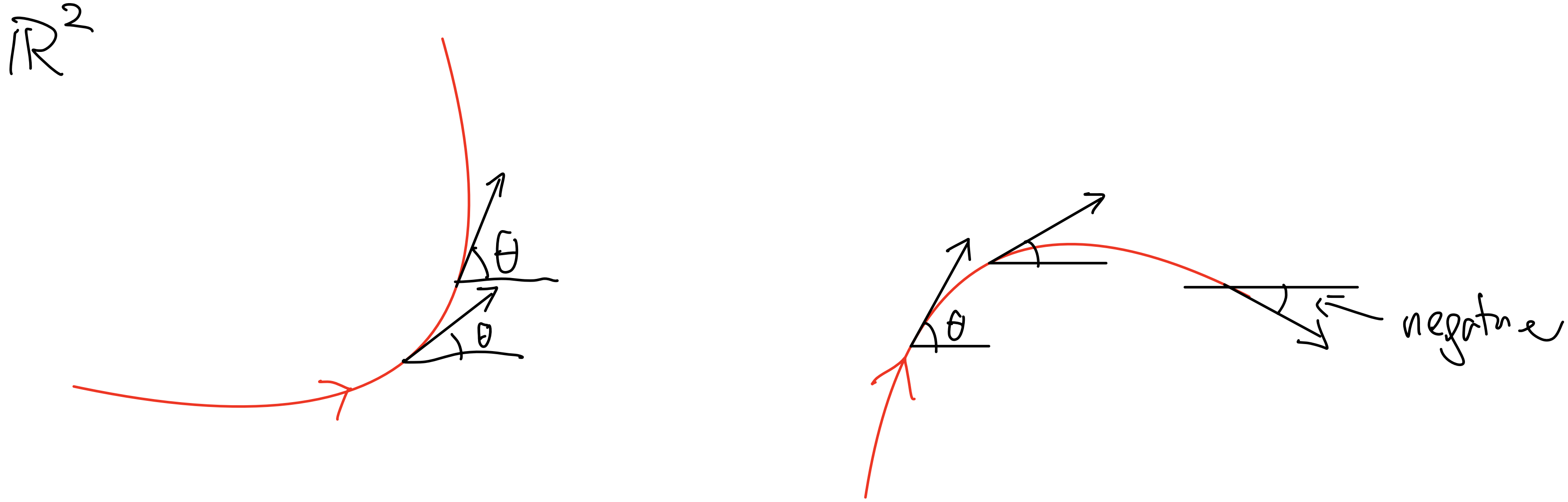
$$= \left|\frac{d\theta}{ds}\right|$$

Definition 2.3.13 (Signed curvature). *Let* $\mathbf{r}(t) = (x(t), y(t))$ *be a regular parametrized curve. The* **signed curvature**, *also denoted by* κ, *of* $\mathbf{r}$ *is*

Kappa

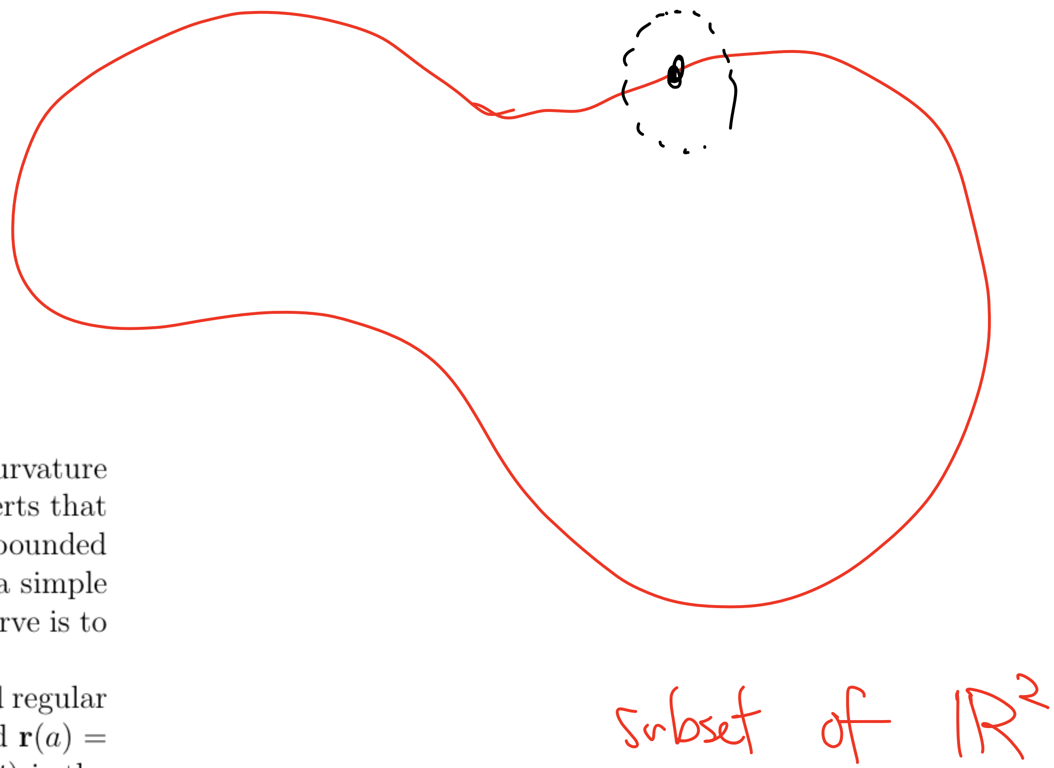
$$\kappa(t) = \frac{d\theta}{ds} = \frac{x'y'' - y'x''}{(x'^2 + y'^2)^{\frac{3}{2}}}$$

where θ *is the angle between the unit tangent vector* $\mathbf{T}$ *and the positive* x*-axis so that* $\mathbf{T} = (\cos\theta, \sin\theta)$.



$$\frac{d\theta}{ds} > 0 \qquad\qquad \frac{d\theta}{ds} < 0$$

Definition 2.3.14 (Simple closed curve). *A regular* **simple closed curve** *in $\mathbb{R}^2$ is a closed and bounded connected subset $C \subset \mathbb{R}^2$ such that for any point $p \in C$, we may find an open set $U_p \subset \mathbb{R}^2$ containing p such that $U_p \cap C$ is the image of a regular parametrized curve.*

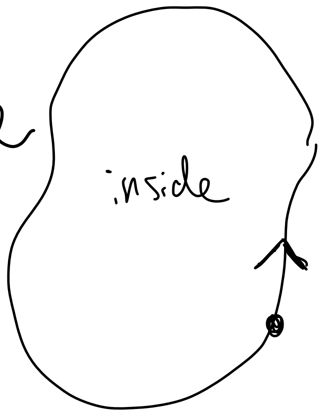


There is a natural orientation which leads to a natural sign of curvature on a regular simple closed curve. The **Jordan curve theorem** asserts that a simple closed curve in $\mathbb{R}^2$ separates the plane into two regions, one bounded and another unbounded. We say that a regular parametrization of a simple closed curve is **positively oriented** if the region bounded by the curve is to the left of the tangent direction.

On a regular simple closed curve, we may find a positively oriented regular parametrization $\mathbf{r}(t)$, $a \leq t \leq b$, such that $\mathbf{r}$ is injective on (a, b) and $\mathbf{r}(a) = \mathbf{r}(b)$. Define a function $\theta(t)$, $a \leq t \leq b$, which is continuous and $\theta(t)$ is the angle between the unit tangent vector $\mathbf{T}(t)$ and the positive x-axis so that $\mathbf{T} = (\cos\theta, \sin\theta)$. The choice of $\theta(t)$ is not unique but any two choices are different by a multiple of 2π. Then since $\mathbf{T}(a) = \mathbf{T}(b)$, the value $\theta(b) - \theta(a)$ must be a multiple of 2π. For regular simple closed curve, we must have $\theta(b) - \theta(a) = 2\pi$.

Theorem 2.3.15. *Let* $\mathbf{r}(t)$, $a \leq t \leq b$, *be a positively oriented regular parametrization of a regular simple closed curve* C *such that* $\mathbf{r}(t)$ *is injective on* (a, b) *and* $\mathbf{r}(a) = \mathbf{r}(b)$. *Let* $\theta(t)$ *be a continuous function such that* $\theta(t)$ *is the angle between the unit tangent vector* $\mathbf{T}(t)$ *and the positive* x*-axis so that* $\mathbf{T} = (\cos\theta, \sin\theta)$. *Then* $\theta(b) - \theta(a) = 2\pi$.

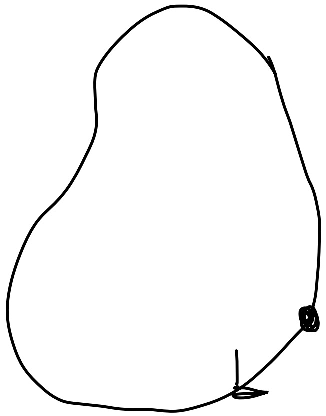
outside



Simple
no self-intersection

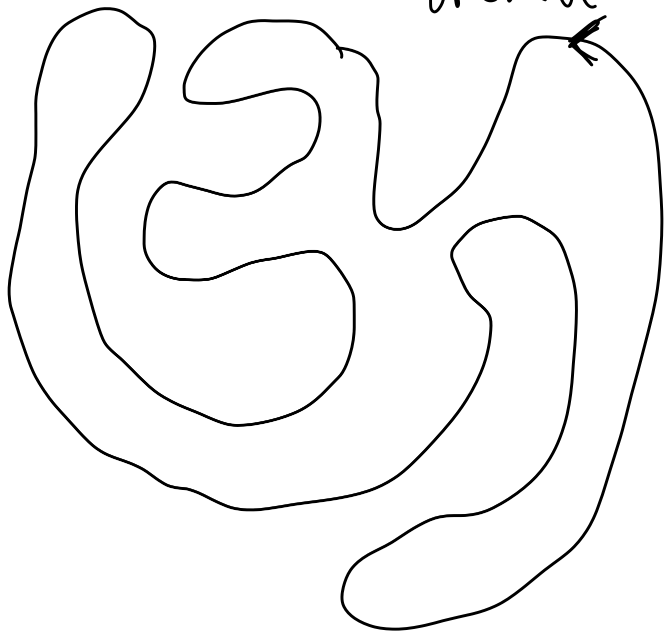
r(a)=r(b)
closed

inside = left hand side
positively oriented

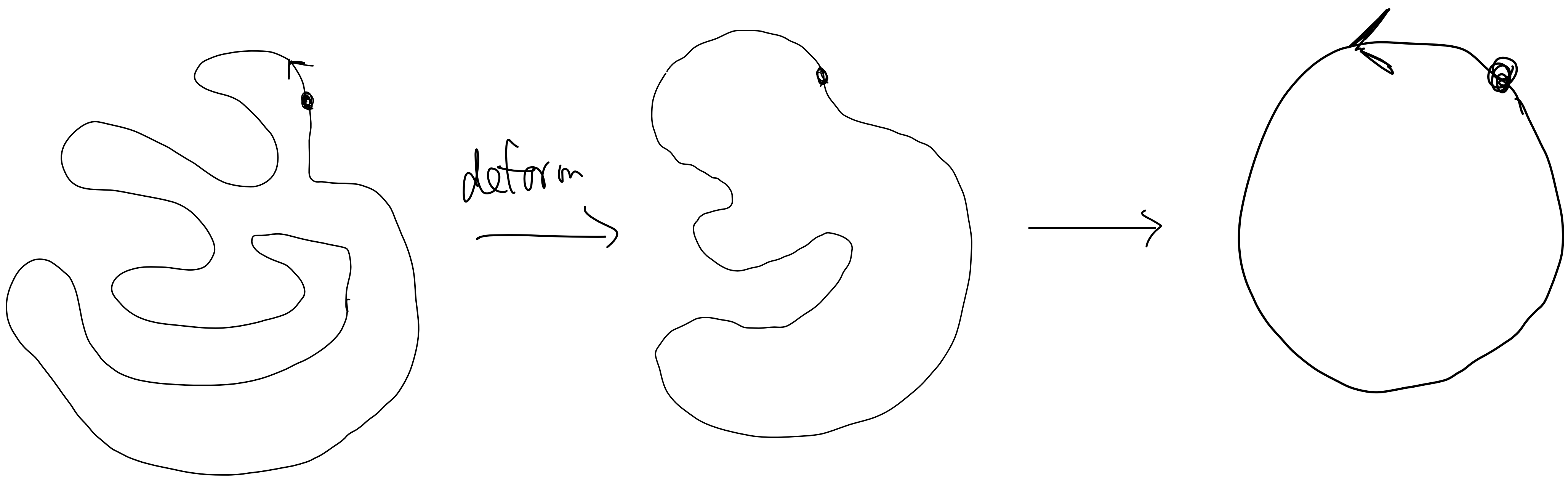


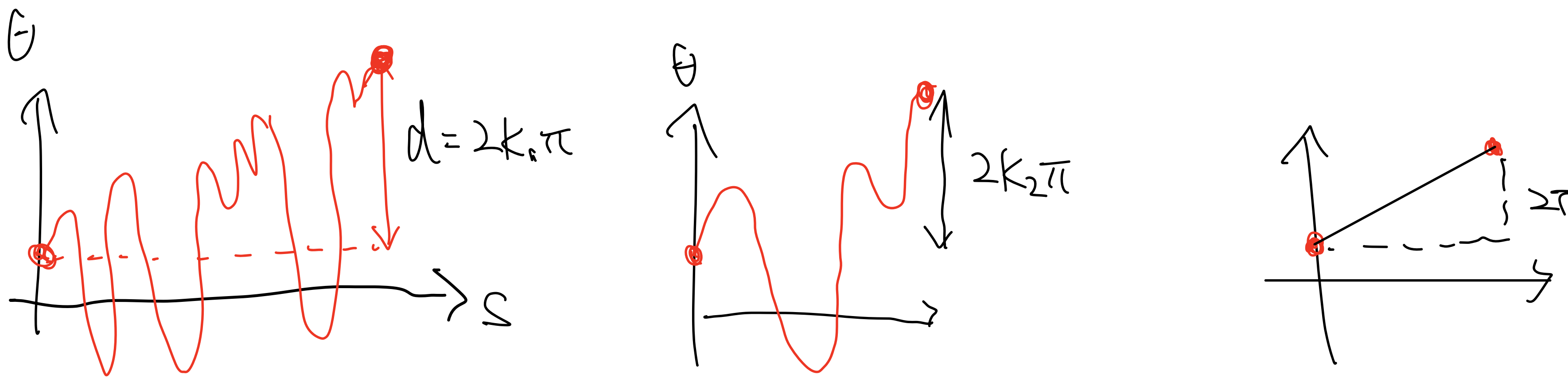
outside = left hand side
negatively oriented

positively
oriented



Deform the simple closed curve to a circle

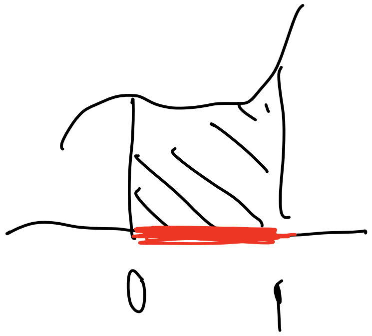




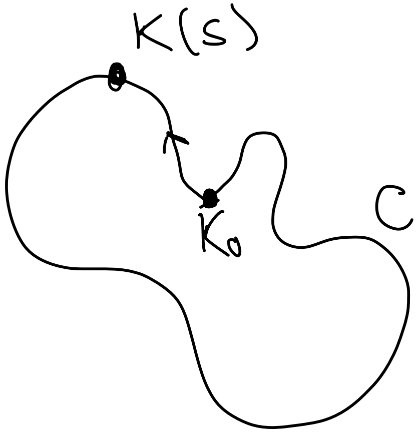
$d = \theta(b) - \theta(a) : 2K_1\pi \longrightarrow 2K_2\pi \longrightarrow 2\pi$ continuously $\Rightarrow K_1 = K_2 = 1$

Theorem 2.3.16. *Let C be a simple closed curve and κ be the signed curvature defined by positively oriented parametrization. Then*

$$\int_C \kappa ds = 2\pi.$$

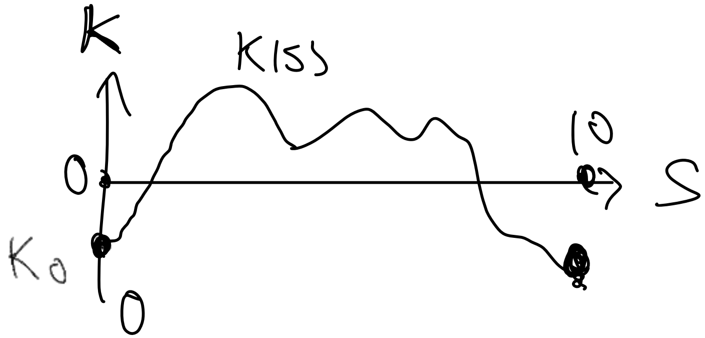


$\int_0^1 f(x)dx = \int_{[0,1]} f(x)dx$



arclength = 10

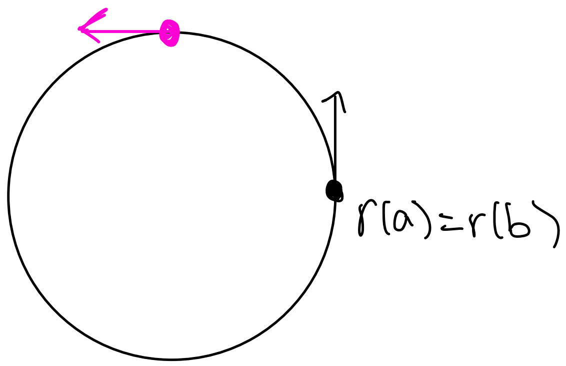
$0 \le s \le 10$

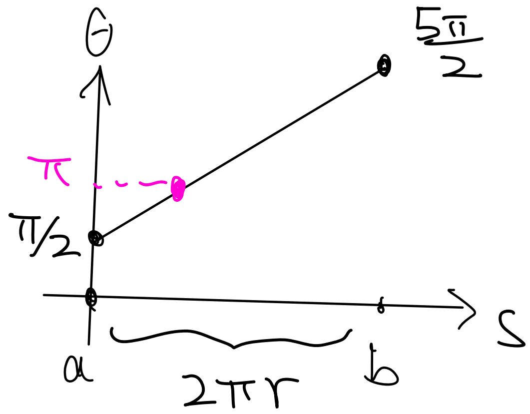


Integral = $\int_C \kappa ds$

Pf $\quad \int_C \kappa ds = \int_C \frac{d\theta}{ds} ds = \int_C d\theta = \left[\theta\right]_a^b = \theta(b) - \theta(a) = 2\pi$

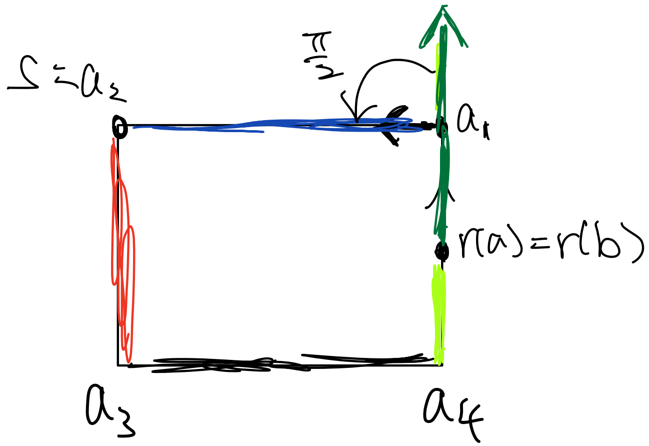
Signed curvature of a simple closed curve can be considered as the continuous version of exterior angles of a polygon. The following theorem is the continuous version of the theorem for sum of exterior angles of polygon.

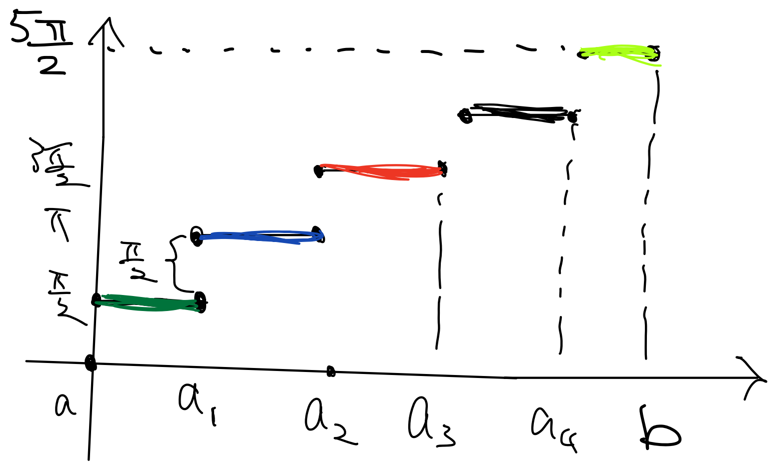




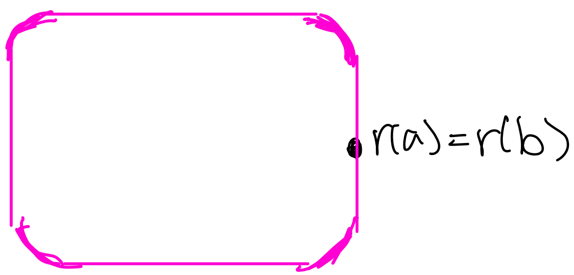
$$\frac{d\theta}{ds} \equiv \frac{\frac{5\pi}{2} - \pi/2}{2\pi r} = \frac{1}{r}$$

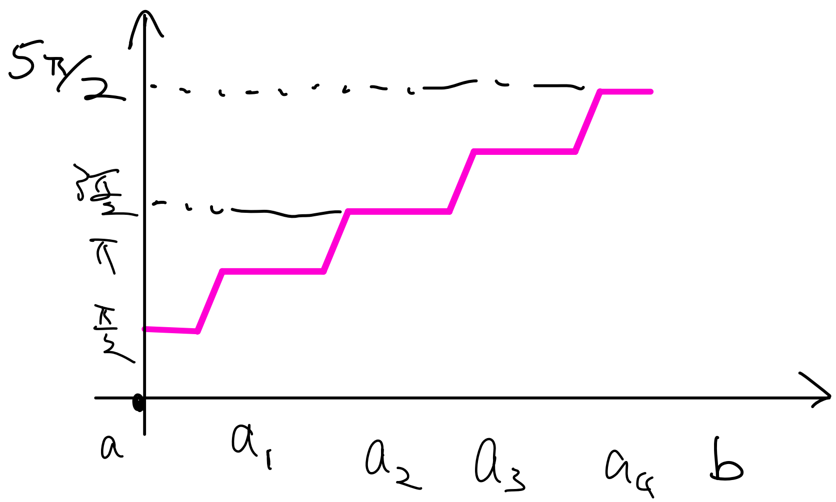
signed curvature





θ discontinuous at vertices





Proposition 2.3.17. *Let* $\mathbf{r}(t)$ *be a regular parametrized curve. Then*

$$\mathbf{a} = \mathbf{r}'' = \frac{dv}{dt}\mathbf{T} + \kappa v^2 \mathbf{N}$$

speed

where $v = \|\mathbf{v}\| = \|\mathbf{r}'\|$.

Proof. First, we have

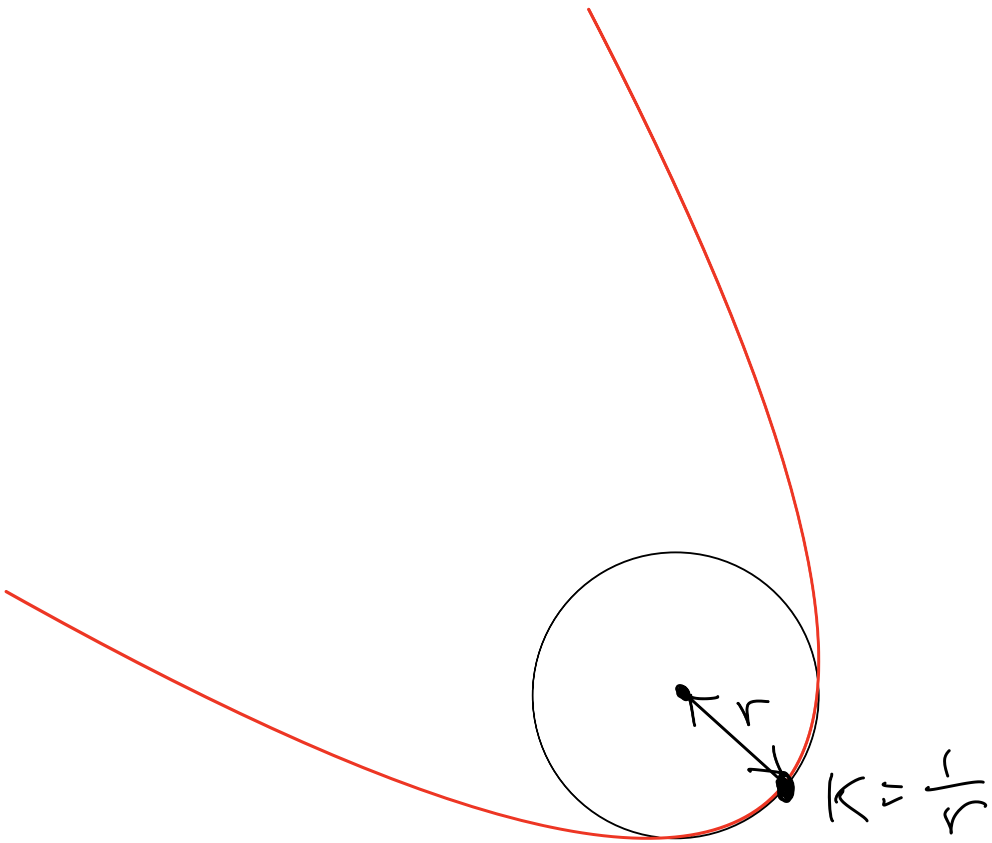
$$\mathbf{r}'(t) = v(t)\mathbf{T}(t).$$

Let s be an arc length parameter, that means $s(t)$ is a function such that $\frac{ds}{dt} = \|\mathbf{r}'(t)\|$. Then $\frac{d}{ds}\mathbf{T} = \kappa\mathbf{N}$ by Theorem 2.3.6 and we have

$$\begin{aligned}
\mathbf{r}'' &= \frac{dv}{dt}\mathbf{T} + v\frac{d}{dt}\mathbf{T} \\
&= \frac{dv}{dt}\mathbf{T} + v\frac{ds}{dt}\frac{d}{ds}\mathbf{T} \\
&= \frac{dv}{dt}\mathbf{T} + \kappa v^2\mathbf{N}.
\end{aligned}$$

Cor $$\kappa(t) = \frac{\langle \mathbf{r}''(t), \mathbf{N}\rangle}{\|\mathbf{r}'(t)\|^2}.$$

There is one more way to interpret the curvature of a curve. When we consider $\mathbf{r}(t)$ as the displacement of a moving particle, we try to find a circle which is closest to the trajectory of the particle at a certain point on the curve. Then the curvature of the curve at that point can be interpreted as the reciprocal of the radius of that circle.



Proposition 2.3.18. *Let* $\mathbf{r}(t)$ *be a regular parametrized curve. Let* $s(t)$ *be an arc length parameter, that is,* $\dfrac{ds}{dt} = \|\mathbf{r}'(t)\|$ *or equivalently* $\left\|\dfrac{d\mathbf{r}}{ds}\right\| = 1$*. Let* $\mathbf{T}$ *and* $\mathbf{N}$ *be the unit tangent and normal vectors, which can be considered as vector valued functions of* t *or* s*, respectively. The curvature* κ *of the curve is characterized by any of the following conditions.*

1. $$\kappa(t) = \frac{\|\mathbf{T}'(t)\|}{\|\mathbf{r}'(t)\|}$$

2. $$\frac{d\mathbf{T}}{ds} = \kappa\mathbf{N}$$

3. *If* $\mathbf{r} = (x, y)$ *is a plane curve, we have*
$$\kappa = \frac{|x'y'' - x''y'|}{(x'^2 + y'^2)^{\frac{3}{2}}}.$$

4. *If* $\mathbf{r} = (x, y, z)$ *is a space curve, we have*
$$\kappa = \frac{\|\mathbf{r}' \times \mathbf{r}''\|}{\|\mathbf{r}'\|^3}.$$

5. $$\kappa = \left\|\frac{d^2\mathbf{r}}{ds^2}\right\|$$

6. *If* $\mathbf{r} = (x, y)$ *is a plane curve and* θ *is the angle between* $\mathbf{T}$ *and the positive* x*-axis, that is,* $\mathbf{T} = (\cos\theta, \sin\theta)$*, then we have*
$$\kappa = \frac{d\theta}{ds}.$$

7. $$\mathbf{r}'' = \frac{dv}{dt}\mathbf{T} + \kappa v^2\mathbf{N}, \;\; \textit{where } v = \|\mathbf{r}'(t)\|.$$